Characterising Weakly Schreier Extensions

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Split extensions of groups

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a diagram in the category of groups.

The diagram is a split extension if

1. k is the kernel of e,
2. e is the cokernel of k,
3. $es = 1$.

Every element $g \in G$ can be written

$$\begin{aligned} g &= g \cdot (se(g^{-1}) \cdot se(g)) \\ &= (g \cdot se(g^{-1})) \cdot se(g). \end{aligned}$$

Notice that $g \cdot se(g^{-1})$ is sent by e to 1.

Thus there exists an $n \in N$ such that $k(n) = g \cdot se(g^{-1})$.

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For each $g \in G$ there exists an $n \in N$ such that $g = k(n) \cdot se(g)$.

Suppose $g = k(n) \cdot s(h)$ and apply e to both sides.

$$\begin{aligned} e(g) &= e(k(n) \cdot s(h)) \\ &= 1 \cdot es(h) \\ &= h. \end{aligned}$$

Thus if $g = k(n) \cdot s(h)$, it must be that $h = e(g)$.

Furthermore if

$$k(n_1) \cdot se(g) = g = k(n_2) \cdot se(g),$$

then $n_1 = n_2$.

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Consider the map $\varphi\colon N \times H \to G$, $\varphi(n,h) = k(n) \cdot s(h)$.

This map is a bijection of sets and so has an inverse φ^{-1}.

$N \times H$ inherits a group structure from φ,

$$(n_1, h_1) \cdot (n_2, h_2) = \varphi^{-1}(\varphi(n_1, h_1)\varphi(n_2, h_2)),$$

turning φ into an isomorphism of groups.

Intuitively $(n_1, h_1) \cdot (n_2, h_2)$ is the element sent by φ to

$$k(n_1) \cdot s(h_1) \cdot k(n_2) \cdot s(h_2).$$

There is an alternative way to view this multiplication.

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The set map $q = \pi_1 \varphi^{-1}$ selects this unique n, which is to say that

$$g = kq(g) \cdot se(g).$$

We can use q to define the following multiplication on $N \times H$

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot q(s(h_1)k(n_2)), h_1 h_2).$$

The map φ will send $(n_1 \cdot q(s(h_1)k(n_2)), h_1 h_2)$ to

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Actions

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a split extension, $\varphi(n,h) = k(n)s(h)$ and $q = \pi_1\varphi^{-1}$.

The map $\alpha(h,n) = q(s(h)k(n))$ is an action of H on N.

An action of H on N is a map $\beta\colon H \to \mathrm{Aut}(N)$.

They corresponds via currying to maps $\alpha\colon H \times N \to N$ satisfying

1. $\alpha(h, n_1n_2) = \alpha(h,n_1)\alpha(h,n_2)$,
2. $\alpha(h_1h_2, n) = \alpha(h_1, \alpha(h_2, n))$,
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Given any action α of H on N

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \alpha(h_1, n_2), h_1 h_2)$$

turns $N \times H$ into a group.

We call the resulting group a semidirect product and write $N \rtimes_\alpha H$.

A semidirect product $N \rtimes_\alpha H$ naturally gives a split extension

$$N \overset{k}{\rightarrowtail} N \rtimes_\alpha H \underset{s}{\overset{e}{\rightleftarrows}} H$$

where $k(n) = (n, 1)$, $e(n, h) = h$ and $s(h) = (1, s)$.

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To what extent did the preceding arguments use group inverses?

Inverses were only used to establish that we can always write

$$g = k(n) \cdot s(h)$$

for unique $n \in N$ and $h \in H$.

Thus the above results apply to any split extension of monoids

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$$

where each g can be written $g = k(n) \cdot s(h)$ for unique n and h.

Such split extensions we call Schreier extensions.

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for unique $n \in N$ and $h \in H$.

Thus the above results apply to any split extension of monoids

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$$

where each g can be written $g = k(n) \cdot s(h)$ for unique n and h.

Such split extensions we call Schreier extensions.

A weaker notion

There exist split extensions of monoids which are not Schreier.

This affords some flexibility not present in the group case.

A weakly Schreier extension is a split extension

$$N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H,$$

in which each $g \in G$ can be written $g = k(n) \cdot s(h)$ for some n and h.

Is there any reason to think that this might be worth studying?

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Topological spaces as monoids

Let X be a topological space and $\mathcal{O}(X)$ its lattice of opens.

There is a natural way to associate a monoid to a topological space.

- $\mathcal{O}(X)$ is closed under binary intersection.
- Since $X \in \mathcal{O}(X)$, the binary intersection has an identity.

Thus $(\mathcal{O}(X), \cap, X)$ is a monoid.

Incidentally this assignment is functorial and has a reflection.

Monoids which behave like lattices of open sets we call frames.

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Artin glueings of topological spaces

Let $N = (|N|, \mathcal{O}(N))$ and $H = (|H|, \mathcal{O}(H))$ be topological spaces.

What topological spaces $G = (|G|, \mathcal{O}(G))$ satisfy that H is an open subspace and N its closed complement?

Such a space G we call an Artin glueing of H by N.

It must be that $|G| = |N| \sqcup |H|$.

Each open $U \in \mathcal{O}(G)$ then corresponds to a pair (U_N, U_H) where $U_N = U \cap N$ and $U_H = U \cap H$.

Thus $\mathcal{O}(G)$ is isomorphic to a frame L_G containing certain pairs (U_N, U_H).

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Artin glueings of topological spaces

For each $U \in \mathcal{O}(H)$ there is a largest open $V \in \mathcal{O}(N)$ such that (V, U) occurs in L_G.

Let $f_G : \mathcal{O}(H) \to \mathcal{O}(N)$ be a function which assigns to each $U \in \mathcal{O}(H)$ the largest V.

This function preserves finite meets.

We have that $(V, U) \in L_G$ if and only if $V \subseteq f(U)$.

Given any finite-meet preserving map $f \colon \mathcal{O}(H) \to \mathcal{O}(N)$ we can construct a frame $\mathrm{Gl}(f)$ as above.

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Artin glueings as weakly Schreier extensions

Let N and H be frames and let $f\colon H \to N$ preserve finite meets.

The following is a split extension of monoids

$$N \overset{k}{\rightarrowtail} \mathrm{Gl}(f) \underset{s}{\overset{e}{\rightleftarrows}} H$$

where $k(n) = (n, 1)$, $e(n, h) = h$ and $s(h) = (f(h), h)$.

Since $(n, h) \in \mathrm{Gl}(f)$ means $n \le f(h)$ we have that

$$\begin{aligned}(n, h) &= (n, 1) \wedge (f(h), h) \\ &= k(n) \wedge s(h).\end{aligned}$$

The diagram is weakly Schreier and it can be shown it's not Schreier.

All weakly Schreier extensions of frames correspond to Artin glueings*

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Weakly Schreier extensions

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a weakly Schreier extension and let $\varphi(n,h) = k(n) \cdot s(h)$.

The map φ is by definition a surjection and so we can quotient by it.

Let E be the equivalence relation given by

$$(n_1, h_1) \sim (n_2, h_2) \iff k(n_1) \cdot s(h_1) = k(n_2) \cdot s(h_2).$$

As in the group case, φ induces a multiplication on $N \times H/E$.

$$[n_1, h_1] \cdot [n_2, h_2] = \overline{\varphi}^{-1}(\overline{\varphi}([n_1, h_1])\overline{\varphi}([n_2, h_2]))$$

Intuitively $[n_1, h_1] \cdot [n_2, h_2]$ is the equivalence class mapped by $\overline{\varphi}$ to

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Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be weakly Schreier, let $\varphi(n,h) = k(n) \cdot s(h)$ and let E be the equivalence relation induced by φ.

A map $q \colon G \to N$ satisfying that for all $g \in G$

$$g = kq(g) \cdot se(g),$$

we call a Schreier retraction.

The class $[n_1 \cdot q(s(h_1)k(n_2)), h_1 h_2]$ is sent by $\overline{\varphi}$ to

$$k(n_1) \cdot s(h_1) \cdot k(n_2) \cdot s(h_2)$$

for any Schreier retraction q.

Thus the multiplication is again determined by a map

$$\alpha(h,n) = q(s(h)k(n)).$$

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Admissible equivalence relations

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\leftrightarrows}} H$ be weakly Schreier, let $\varphi(n,h) = k(n) \cdot s(h)$ and let E be the equivalence relation induced by φ.

The equivalence relation E satisfies the following properties.

1. $(n_1, 1) \sim (n_2, 1)$ implies $n_1 = n_2$,
2. $(n_1, h_1) \sim (n_2, h_2)$ implies $h_1 = h_2$,
3. $(n_1, h) \sim (n_2, h)$ implies $(nn_1, h) \sim (nn_2, h)$,
4. $(n_1, h) \sim (n_2, h)$ implies $(n_1, hh') \sim (n_2, hh')$.

Suppose h has a right inverse h^*.

- $(n_1, h) \sim (n_2, h)$ implies $(n_1, hh^*) \sim (n_2, hh^*)$.
- $(n_1, 1) \sim (n_2, 1)$ implies $n_1 = n_2$.

Thus for a group the quotient must always be discrete.

Any equivalence relation satisfying the above we call admissible.

Admissible equivalence relations

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be weakly Schreier, let $\varphi(n,h) = k(n) \cdot s(h)$ and let E be the equivalence relation induced by φ.

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Thus for a group the quotient must always be discrete.

Any equivalence relation satisfying the above we call admissible.

Admissible equivalence relations

Let $N \overset{k}{\rightarrowtail} G \overset{e}{\underset{s}{\rightleftarrows}} H$ be weakly Schreier, let $\varphi(n,h) = k(n) \cdot s(h)$ and let E be the equivalence relation induced by φ.

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Characterizing weakly Schreier extensions

Let E be an admissible equivalence relation on $N \times H$ and let α be a compatible action.

Theorem
The set $N \times H/E$ equipped with multiplication

$$[n_1, h_1] \cdot [n_2, h_2] = [n_1\alpha(h_1, n_2), h_1h_2],$$

is a monoid.

Theorem
The diagram

$$N \overset{k}{\rightarrowtail} N \times H/E \underset{s}{\overset{e}{\rightleftarrows}} H$$

where $k(n) = [n, 1]$, $e([n, h]) = h$ and $s(h) = [1, h]$, is a weakly Schreier extension.

The processes described in this talk are inverses.

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Constructing Examples

Let N be a monoid and H a monoid with no non-trivial left units.

Consider the quotient Q on $N \times H$ given by

$$(n, h) \sim (n', h) \text{ for all } n \in N \text{ and } 1 \neq h \in H.$$

This quotient is admissible and can be identified with $N \sqcup (H - \{1\})$ where

- $[n, 1] \mapsto n$
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Every function $\alpha\colon N \times H \to N$ is compatible with the quotient.

Recall that $[n, h] \cdot [n', h'] = [n \cdot \alpha(h, n'), hh']$.

Because of the quotient $n \cdot \alpha(h, n')$ is irrelevant whenever $hh' \neq 1$.

When $hh' = 1$ this means $h = 1$ and so $n\alpha(h, n') = nn'$.

Thinking in terms of $N \sqcup (H - \{1\})$ multiplication becomes

- $n \cdot n'$ the usual product in N,
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The coarsest quotient

Can we relax the condition that H contain no left units?

Consider the quotient whereby

- $(n, h) \sim (n', h)$ if h is not a left unit,
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When does there exist a compatible action?

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- $(n, h) \sim (n', h)$ if h is not a left unit,
- $(n, h) \sim (n', h) \Rightarrow n = n'$ if h is a left unit.

This is the coarsest admissible quotient on $N \times H$.

When does there exist a compatible action?

Compatible actions

Let N and H be monoids and Q the coarsest admissible quotient.

The set $L(H)$ of left-units of H forms a submonoid of H.

The complement $\overline{L(H)}$ of $L(H)$ forms a right ideal.

- It is clear it is closed under multiplication.
- If $x \in \overline{L(H)}$ and $h \in H$, then $(xh)y = 1$ implies that hy is a right inverse of x.

Theorem
*If $\overline{L(H)}$ is a two-sided ideal, each map $\alpha : H \times N \to N$ in which $\alpha|_{L(H)\times N}$ is an action of $L(H)$ on N, is compatible with the coarse quotient Q. Otherwise, no map α is compatible with Q.**

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Intuition

Let N and H be monoids and Q the coarsest admissible quotient.

To illustrate the general idea lets looks at this requirement

$$[\alpha(hh', n), hh'] = [\alpha(h, \alpha(h', n)), hh'].$$

If hh' is a left unit then $\alpha(hh', n) = \alpha(h, \alpha(h', n))$.

Additionally h and h' must be left units as well.

So with respect to this requirement (and all others) α behaves like an action of $L(H)$ on N.

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Otherwise, if hh' is not a left unit then things almost work completely as most of the requirements are immediately satisfied.

However suppose $\overline{L(H)}$ is not a two-sided ideal.

Then there exists $x \in L(H)$ and $y \in \overline{L(H)}$ such that $xy \in L(H)$.

The requirement

$$(n, y) \sim (n', y) \text{ implies } [\alpha(x, n), xy] = [\alpha(x, n'), xy]$$

gives that $\alpha(x, n) = \alpha(x, n')$ for all $n, n' \in N$.

We also know that $\alpha(x, 1) = 1$ and so $\alpha(x, n) = 1$ for all $n \in N$

Finally consider
$[n, 1] = [\alpha(1, n), 1] = [\alpha(xx^{-1}, n), xx^{-1}] = \alpha(x, \alpha(x^{-1}, n), 1) = [1, 1]$.

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Examples

This two sided property holds whenever H is

- H is finite,
- H is commutative,
- H is a group,
- H has no inverses at all
- H is a monoid of $n \times n$ matrices over a field

The result can be generalised where $\overline{L(H)}$ is replaced with any prime ideal.

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